Geodesic Active Fields

A Geometric Framework for Image Registration

Dominique Zosso[1], Xavier Bresson[2], Jean-Philippe Thiran[1]

[1] Signal Processing Laboratory (LTS5), Ecole Polytechnique Fédérale de Lausanne (EPFL), Switzerland
[2] Department of Computer Science, City University of Hong Kong.

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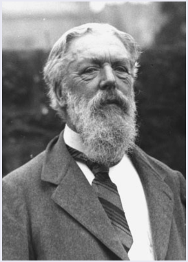
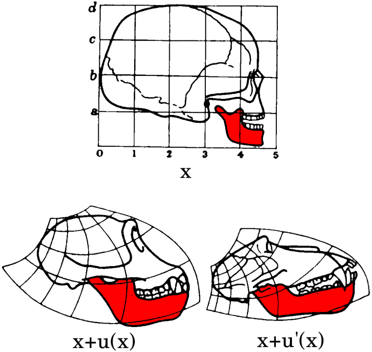
A Brief on Variational Image Registration

On Growth and Form



D'Arcy Wentworth Thompson
(∗1860 – †1948)



Energy minimization

$$\overset{*}{\mathbf{u}} = \underset{\mathbf{u}}{argmin}\{E(\mathbf{u})\}$$

$$\text{e.g.} \quad E = \int_\Omega (M(\mathbf{x}+\mathbf{u}) - F(\mathbf{x}))^2 \, \mathrm{d}\mathbf{x}$$

... with regularization

$$E = \int_\Omega (M(\mathbf{x}+\mathbf{u}) - F(\mathbf{x}))^2 \, \mathrm{d}\mathbf{x} + \alpha \cdot \int_\Omega |\nabla \mathbf{u}(\mathbf{x})| \mathrm{d}\mathbf{x}$$

More general:

$$E = E_{\text{data}} + \alpha \cdot E_{\text{regularization}}, \quad \alpha > 0.$$

The Beltrami Framework

N. Sochen, R. Kimmel, and R. Malladi, "A General Framework For Low Level Vision," *IEEE TIP*, 7(**3**):310–318, 1998

1. An Embedding
2. A Riemannian structure
3. A Measure on the embedding map

2D grayscale Image

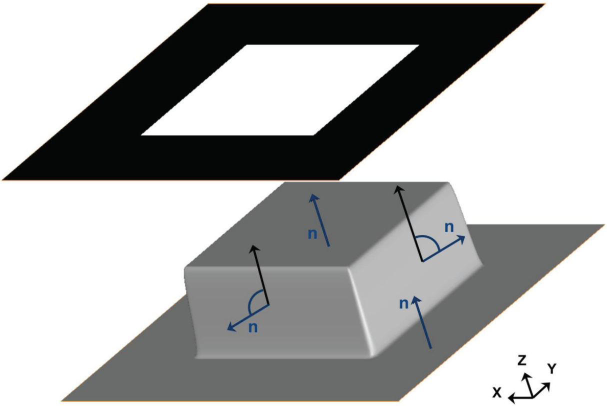
$$\begin{aligned} I : (x, y) \in \Omega &\mapsto I(x, y) \in \mathbb{R} \\ \text{(space)} &\mapsto \text{(feature)} \end{aligned}$$

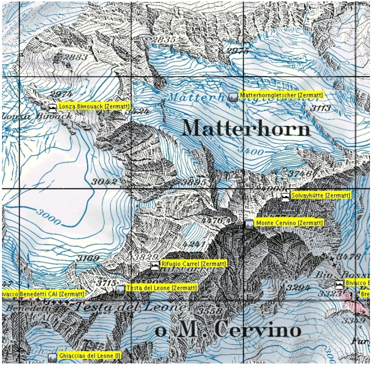
Beltrami embedding

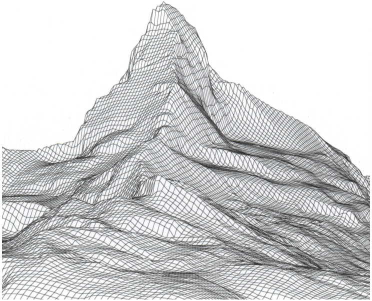
$$\begin{aligned} X : (x, y) \in \Omega &\mapsto (x, y, I(x, y)) \in \Omega \times \mathbb{R} \\ \text{(space)} &\mapsto \text{(space,feature)} \end{aligned}$$

More general

$$X : \Sigma \mapsto M; \quad X : (\sigma^1 \dots \sigma^n) \mapsto (X^1(\sigma^1 \dots \sigma^n) \dots X^m(\sigma^1 \dots \sigma^n))$$


n
n
n
n
Z
Y
X


2835
2975
2883
2974
Lonza Biwouack [Zermatt]
Matterhorngletscher [Zermatt]
3113
Matterhorn
3400
3042
3746
Solvayhütte [Zermatt]
3000
Monte Cervino [Zermatt]
3169
3829
Rifugio Carrel [Zermatt]
3478
3715
Testa del Leone [Zermatt]
3294
Testa del Leone
3558
M. Cervino
Ghiacciao del Leone []



Σ and M are isometric

Let h_{ij} be the metric of M, e.g.

$$h_{ij} = \text{arbitrary} \qquad\qquad h_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta^2 \end{bmatrix}$$

For $g_{\mu\nu}$ on Σ the *pullback relation* yields

$$g_{\mu\nu} = h_{ij}\partial_\mu X^i \partial_\nu X^j \qquad\qquad g_{\mu\nu} = \begin{bmatrix} 1+\beta^2 I_x^2 & \beta^2 I_x I_y \\ \beta^2 I_x I_y & 1+\beta^2 I_y^2 \end{bmatrix}$$

$$g = \det(g_{\mu\nu}) \qquad\qquad g = 1+\beta^2\|\nabla I\|^2$$

Polyakov Action

$$S = \int \sqrt{g}\,\mathrm{d}x\mathrm{d}y \qquad\qquad S = \int \sqrt{1 + \beta^2 \|\nabla I\|^2}\,\mathrm{d}x\mathrm{d}y$$

Beltrami Flow

$$\partial_t X^i = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\,g^{\mu\nu}\partial_\nu X^i\right) + \Gamma^i_{jk}\partial_\mu X^j \partial_\nu X^k g^{\mu\nu}$$

$$\Gamma^i_{jk} = \frac{1}{2}h^{il}\left(\partial_j h_{kl} + \partial_k h_{jl} - \partial_l h_{jk}\right)$$

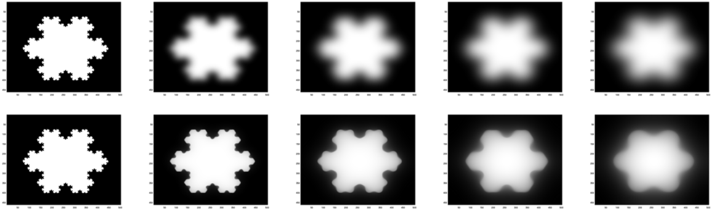
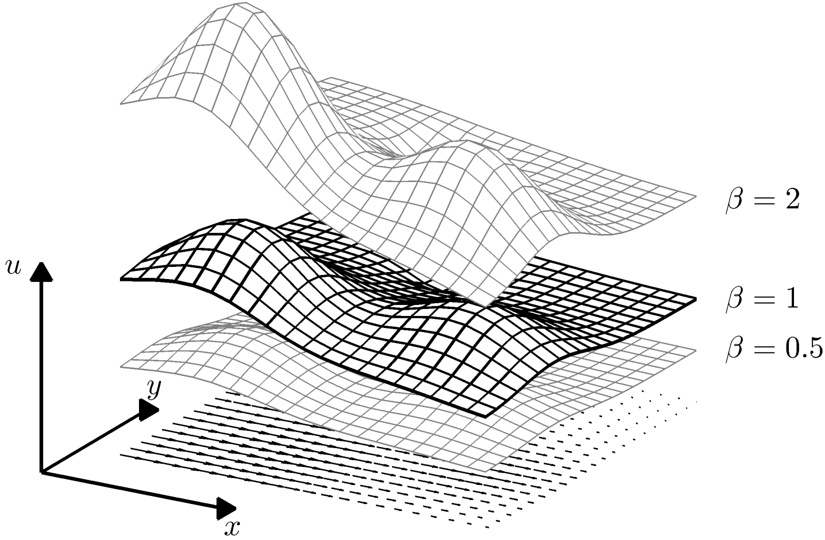


Image Registration + Beltrami Framework


$\beta = 2$
$\beta = 1$
$\beta = 0.5$
u
y
x

Beltrami Regularization

$$X : (x, y) \mapsto (x, y, u, v)$$

$$h_{ij} = \mathrm{diag}(1, 1, \beta^2, \beta^2)$$

$$g_{\mu\nu} = h_{ij} \partial_\mu X^i \partial_\nu X^j$$

$$E = \int_\Omega (M(\mathbf{x} + \mathbf{u}) - F(\mathbf{x}))^2 \, \mathrm{d}\mathbf{x} + \alpha \cdot \int \sqrt{g} \mathrm{d}\mathbf{x}, \quad \alpha > 0.$$

DOH!

- R. Ben-Ari and N. Sochen, "Non-isotropic regularization of the correspondence space in stereo-vision," in **ICPR 2004**, vol. 4, Aug. 2004, pp. 293–296.
- —, "A geometric approach for regularization of the data term in stereo-vision," **J. Math. Imaging Vis.**, vol. 31, no. 1, pp. 17–33, May 2008.
- —, "A geometric framework and a new criterion in optical flow modeling," **J. Math. Imaging Vis.**, vol. 33, no. 2, pp. 178–194, Feb. 2009.

Geodesic Active Fields

$$X : (x, y) \mapsto (x, y, u, v)$$

$$h_{ij} = \mathrm{diag}(1, 1, \beta^2, \beta^2)$$

$$g_{\mu\nu} = h_{ij} \partial_\mu X^i \partial_\nu X^j$$

$$E = \int (1 + \alpha \cdot f) \sqrt{g} \mathrm{d}\mathbf{x}$$

$$\text{e.g. } f = (M(\mathbf{x} + \mathbf{u}) - F(\mathbf{x}))^2$$

... and its Flow

$$\partial_t X^i = (1 + \alpha \cdot f) H^i + \alpha \partial_k f g^{\mu\nu} \partial_\mu X^k \partial_\nu X^i - \alpha \frac{m \cdot n}{2} \partial_k f h^{ki},$$

$$H^i = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu X^i \right) + \Gamma^i_{jk} \partial_\mu X^j \partial_\nu X^k g^{\mu\nu}$$

A Splitting Scheme

Discrete samples:

$$E_{GAF} = \int f\sqrt{g}\,\mathrm{d}\mathbf{x} \approx \sum_{i=1}^{N} F_i G_i = F^T G$$

$$F, G \in \mathbb{R}^N$$

Let's split:

$$\min_{\mathbf{u}} \left\{ F(\mathbf{u})^T G(\mathbf{u}) \right\} \;\Leftrightarrow\; \min_{\mathbf{u},\mathbf{v}} \left\{ F(\mathbf{u})^T G(\mathbf{v}) \right\} \quad s.t. \;\; \mathbf{u} = \mathbf{v}$$

$$\mathbf{u}, \mathbf{v} \in \mathbb{R}^{N \times p}$$

Augmented Lagrangian:

$$\Lambda(\mathbf{u}, \mathbf{v}, \lambda) = F(\mathbf{u})^T G(\mathbf{v}) + \lambda \cdot (\mathbf{u} - \mathbf{v}) + \frac{r}{2}\|\mathbf{u} - \mathbf{v}\|^2$$

$$\lambda \in \mathbb{R}^{N \times p} \qquad r \in \mathbb{R}^+$$

Concept:

$$\left\{\begin{array}{lcl} \mathbf{u}^{k+1} & = & \arg\min\limits_{\mathbf{u}} \left\{ F(\mathbf{u})^T G(\mathbf{v}^k) + \lambda^k \cdot (\mathbf{u} - \mathbf{v}^k) + \frac{r}{2}\|\mathbf{u} - \mathbf{v}^k\|^2 \right\}, \\ \mathbf{v}^{k+1} & = & \arg\min\limits_{\mathbf{v}} \left\{ F(\mathbf{u}^{k+1})^T G(\mathbf{v}) + \lambda^k \cdot (\mathbf{u}^{k+1} - \mathbf{v}) + \frac{r}{2}\|\mathbf{u}^{k+1} - \mathbf{v}\|^2 \right. \\ \lambda^{k+1} & = & \lambda^k + r(\mathbf{u}^{k+1} - \mathbf{v}^{k+1}), \end{array}\right.$$

Sketch of a solution:

$$\left\{\begin{array}{lcl} \mathbf{u}^{k+1} & = & \mathbf{v}^k - \frac{1}{r}\left(\lambda^k + \mathrm{diag}(G(\mathbf{v}^k)) \cdot \frac{\partial F}{\partial \mathbf{u}}(\mathbf{u}^k)\right) \\ \mathbf{v}^{k+1} & = & \left(I - \frac{1}{r}\,\mathrm{diag}(F(\mathbf{u}^{k+1})) \cdot W\right)^{-1}\left(\mathbf{u}^{k+1} + \frac{1}{r}\lambda^k\right), \\ \lambda^{k+1} & = & \lambda^k + r(\mathbf{u}^{k+1} - \mathbf{v}^{k+1}), \end{array}\right.$$

Some Results

Stereo Vision

$$X : (x, y) \mapsto (x, y, u)$$

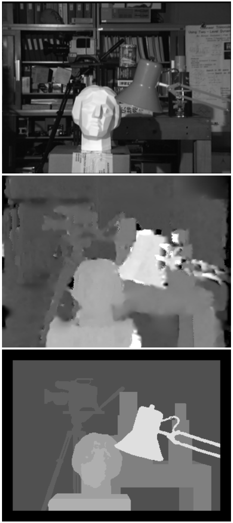
$$h_{ij} = \mathrm{diag}(1, 1, \beta^2)$$

$$g_{\mu\nu} = h_{ij}\partial_\mu X^i \partial_\nu X^j$$

Absolute error (L^1-norm):

$$f = \sqrt{(M(\mathbf{x} + \mathbf{u}) - F(\mathbf{x}))^2 + \epsilon^2}$$

$$E = \int (1 + \alpha \cdot f)\sqrt{g}\,\mathrm{d}\mathbf{x}$$



2D Medical Imaging

$$X : (x, y) \mapsto (x, y, \color{red}{u, v})$$

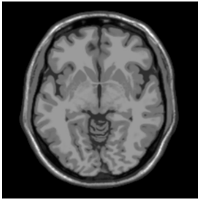
$$h_{ij} = \mathrm{diag}(1, 1, \color{red}{\beta^2, \beta^2})$$

$$g_{\mu\nu} = h_{ij} \partial_\mu X^i \partial_\nu X^j$$

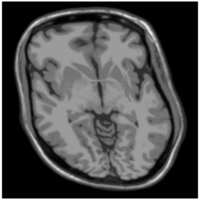
Squared error (L^2-norm):

$$f = (M(\mathbf{x} + \mathbf{u}) - F(\mathbf{x}))^2$$

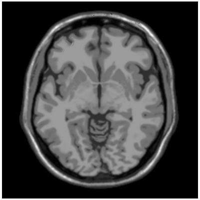
$$E = \int (1 + \alpha \cdot f) \sqrt{g} \mathrm{d}\mathbf{x}$$



F



M



$M \circ T$

Multimodal Medical Imaging

$$X : (x, y) \mapsto (x, y, u, v)$$

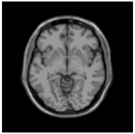
$$h_{ij} = \mathrm{diag}(1, 1, \beta^2, \beta^2)$$

$$g_{\mu\nu} = h_{ij}\partial_\mu X^i \partial_\nu X^j$$

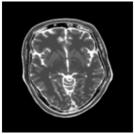
Local, joint entropy (MI):

$$f = -\ln\left(p^{fm}(F(\mathbf{x}), M(\mathbf{x}+\mathbf{u}))\right)$$

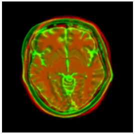
$$E = \int (1 + \alpha \cdot f)\sqrt{g}\,\mathrm{d}\mathbf{x}$$



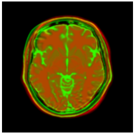
(a) F



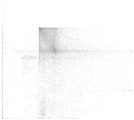
(b) M



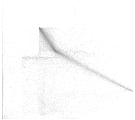
(c) $F|M$



(d) $F|M \circ T_0$



(e) p^{fm}



(f) $p^{f\{m \circ T_0\}}$

Gaussian Scale-Space

$$X : (x, y, \sigma) \mapsto (x, y, \sigma, u, v)$$

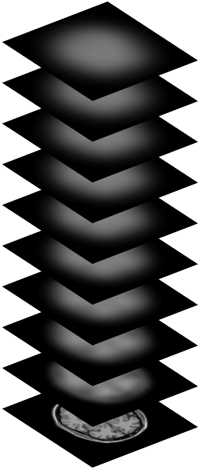
$$h_{ij} = \frac{1}{\sigma^2} \operatorname{diag}(1, 1, 1, \beta^2, \beta^2)$$

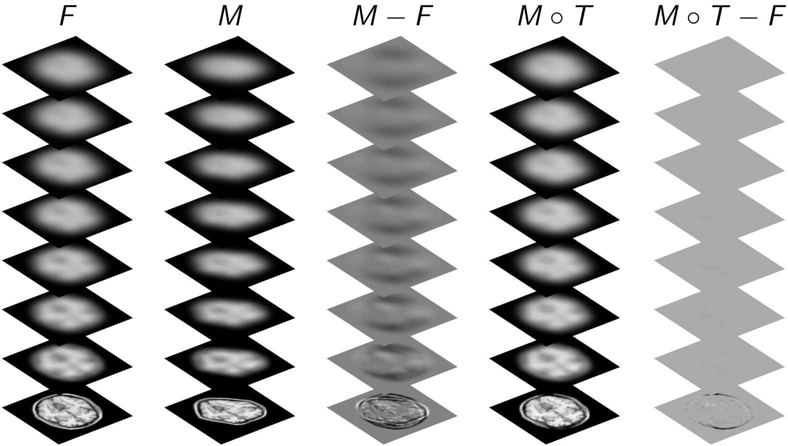
$$g_{\mu\nu} = h_{ij} \partial_\mu X^i \partial_\nu X^j$$

Squared error (L^2-norm):

$$f = (M(\mathbf{x} + \mathbf{u}) - F(\mathbf{x}))^2$$

$$E = \int (1 + \alpha \cdot f) \sqrt{g} \mathrm{d}\mathbf{x}$$




F
M
$M - F$
$M \circ T$
$M \circ T - F$

Summary

Geodesic Active Fields:

- Embedding:

$$X : (x, y) \mapsto (x, y, u)$$

- Metrics:

$$h_{ij} = \mathrm{diag}(1, 1, \beta^2),$$

$$g_{\mu\nu} = h_{ij} \partial_\mu X^i \partial_\nu X^j$$

- Energy:

$$E_{GAF} = \int (1 + \alpha \cdot f) \sqrt{g} \mathrm{d}\mathbf{x}$$

A novel method called geodesic active fields to register images:

- Standard Cartesian images, images on arbitrary Riemannian manifolds
- Parametrization invariance
- Geometric regularization: interpolate TV and Gaussian
- Data-dependent spatially adaptive regularization

Outlook

- Faster numerical schemes
- Diffeomorphic deformations
- etc.

$$S = \int f \sqrt{g} \mathrm{d}\sigma$$

Task	Emb. features	f	e.g.
Intensity*	$I \ / \ R, G, B$	1	Sochen1998
Texture	G	1	Kimmel2000
Deformation field	u	1	BenAri2004/
regularization	I, u	1	2008/2009
Segmentation	C	$\frac{1}{1+\lambda\|\nabla I\|^2}$	Bresson2006
Registration	u	$1 + \alpha \cdot d(M', F)$	Zosso2010
	$I, u \ / \ G, u$	$1 + \alpha \cdot d(M', F)$	-

* Scale-space and regularization (denoising).

It has not escaped our notice that the specific weighted embedding we have postulated immediately suggests similar usage in other computer vision tasks.

Related publications

Tech reports:

- D. Zosso, X. Bresson and JP. Thiran, "Geodesic Active Fields – A Geometric Framework for Image Registration," **Technical Report**, LTS-2010-001, and **CAM Report 10-14**, Jan. 2010.

Journals:

- D. Zosso, X. Bresson and JP. Thiran, "Geodesic Active Fields – A Geometric Framework for Image Registration," submitted to **IEEE Trans Image Process**, Mar. 2010.

Conferences:

- D. Zosso and JP. Thiran, "Geodesic Active Fields on the Sphere," **ICPR 2010**, Istanbul, Turkey, Aug. 2010.
- D. Zosso, X. Bresson and JP. Thiran, "Geodesic Active Fields – A Geometric Framework for Image Registration," **ICNAAM 2010**, Rhodes, Greece, Sep. 2010.